Practices before the class (March 24)

- **(T/F)** If A is similar to a diagonalizable matrix B, then A is also diagonalizable.
- **(T/F)** Similar matrices always have exactly the same eigenvalues.
- **(T/F)** Similar matrices always have exactly the same eigenvectors.

Practices before the class (March 24)

- **(T/F)** If A is similar to a diagonalizable matrix B, then A is also diagonalizable.
  True.
  If $B = PDP^{-1}$, where D is a diagonal matrix, and if $A = QBQ^{-1}$, then
  $A = Q\left(PDP^{-1}\right)Q^{-1} = (QP)D(QP)^{-1}$, which shows that A is diagonalizable.
- **(T/F)** Similar matrices always have exactly the same eigenvalues.
  True. This follows from Theorem 4 in Section 5.2.
  Recall Theorem 4: Similar matrices have the same characteristic polynomial and hence the same eigenvalues.

Practices before the class (March 24)

- **(T/F)** Similar matrices always have exactly the same eigenvectors.
  False. We can refer this as a general fact in the future. One counter-example can be constructed below. Recall Example 1 in Section 5.2, where $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$. By the computation in the notes, A is similar to $D = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$. The eigenvectors for A are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, corresponding to the eigenvalues 2 and 8, respectively. But the eigenvectors for D can be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,respecting to the eigenvalues 2 and 8.

5.5 Complex Eigenvalues

Review of Complex Numbers

Solutions to $x^2+3=0$?

A **complex number** is a number written in the form

$$z = a + bi$$

where a and b are real numbers and i is a formal symbol satisfying the relation $i^2 = -1$. We can take $i = \sqrt{-1}$

- The number a is the **real part** of z, denoted by $\mathrm{Re}\, z$,
- and b is the **imaginary part** of z, denoted by $\mathrm{Im}\, z$. Note $\mathrm{Im}\, z = b$, which is a real number.
- Two complex numbers are considered equal if and only if their real and imaginary parts are equal.
- The **conjugate** of $z = a + bi$ is the complex number $\bar{z}$ (read as " z bar"), defined by

$$\bar{z} = a - bi$$

Example 1. Find all real and complex solutions to the equation $x^4 + 6x^2 + 9 = 0$

$(x^2)^2 + 6x^2 + 9 = 0 \Rightarrow (x^2+3)^2 = 0 \Rightarrow x^2 + 3 = 0$

$\Rightarrow x^2 = -3 \Rightarrow x = \pm\sqrt{-3} = \pm\sqrt{3}\cdot\sqrt{-1} = \pm\sqrt{3}\, i$

Thus $x = \pm\sqrt{3}\, i$, each of them has multiplicity 2.

Example 2. Find all real and complex eigenvalues of the matrice

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 5 & -5 \end{bmatrix}$$

ANS: We solve the characteristic equation $|A - \lambda I| = 0$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -1-\lambda & -1 \\ 0 & 5 & -5-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & -1 \\ 5 & -5-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(\lambda+1)(\lambda+5)+5\right]$$

$$= (2-\lambda)(\lambda^2+6\lambda+10) = 0$$

Thus $2-\lambda = 0 \Rightarrow \lambda = 2$

or $\lambda^2 + 6\lambda + 10 = 0 \Rightarrow \lambda = \frac{-6\pm\sqrt{6^2-4\times10}}{2} = \frac{-6\pm\sqrt{-4}}{2} = -3\pm i$

Thus $\lambda = 2$, $\lambda = -3+i$ and $\lambda = -3-i$

Real and Imaginary Parts of Vectors

$\in \mathbb{C}^n$

The real and imaginary parts of a complex vector $\mathbf{x}$ are the vectors $\operatorname{Re}\mathbf{x}$ and $\operatorname{Im}\mathbf{x}$ in $\mathbb{R}^n$ formed from the real and imaginary parts of the entries of $\mathbf{x}$. Thus,

$\vec{x} \in \mathbb{C}^3$

$$\mathbf{x} = \operatorname{Re}\mathbf{x} + i \operatorname{Im}\mathbf{x}$$

conjugate of $\vec{x}$.

Eg: $\vec{x} = \begin{bmatrix} 2+i \\ 3 \\ 5+2i \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Then $\operatorname{Re}\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $\operatorname{Im}\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. $\overline{\vec{x}} = \begin{bmatrix} 2-i \\ 3 \\ 5-2i \end{bmatrix}$

Eigenvalues and Eigenvectors of a Real Matrix That Acts on $\mathbb{C}^n$

Let A be an $n \times n$ matrix whose entries are real.

Then $\overline{A\mathbf{x}} = \bar{A}\overline{\mathbf{x}} = A\overline{\mathbf{x}}$.

If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector in $\mathbb{C}^n$, then

$$\underline{A\overline{\mathbf{x}}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \underline{\bar{\lambda}\overline{\mathbf{x}}}$$

Remark: Thus $\bar{\lambda}$ is also an eigenvalue of A, with $\overline{\mathbf{x}}$ a corresponding eigenvector. This shows that when A is real, its complex eigenvalues and eigenvectors occur in conjugate pairs. (We will use this fact to simplify the computation in **Example 3**.)

$\mathbb{C}^2 \xrightarrow{A} \mathbb{C}^2$

Example 3. Let the given matrix ~~is~~ acts on $\mathbb{C}^2$. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$.

$A = \begin{bmatrix} -3 & -1 \\ 2 & -5 \end{bmatrix}$

ANS: The characteristic equation is $|A-\lambda I| = 0$:

$$\begin{vmatrix} -3-\lambda & -1 \\ 2 & -5-\lambda \end{vmatrix} = (\lambda+3)(\lambda+5)+2 = \lambda^2+8\lambda+17 = 0$$

So the eigenvalues of A are

$$\lambda = \frac{-8 \pm \sqrt{8^2-4\times 17}}{2} = \frac{-8\pm\sqrt{-4}}{2} = -4 \pm i.$$

For $\lambda_1 = -4+i$: we solve $(A-\lambda I)\vec{x} = \vec{0}$. The augmented matrix:

$$\begin{bmatrix} A-(-4+i) & \vec{0} \end{bmatrix} = \begin{bmatrix} 1-i & -1 & 0 \\ 2 & -1-i & 0 \end{bmatrix} \sim \begin{bmatrix} 1-i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that $R1 \times (1+i) = R2$. So the two rows implies the same eqn:

$$2x_1 + (-1-i)\underset{\text{free}}{x_2} = 0$$

Thus $\begin{cases} x_1 = \frac{1}{2}(1+i)x_2 \\ x_2 \text{ is free} \end{cases}$, $\vec{x} = \begin{bmatrix} \frac{1}{2}(1+i)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix}$

We can choose an eigenvector for $\lambda_1 = -4+i$ to be

$$\vec{v}_1 = 2 \cdot \begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

For $\lambda_2 = -4-i$: From the remark above Example 3.

We know $A\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$, which implies

$A\overline{\vec{v}_1} = \overline{\lambda}_1 \overline{\vec{v}_1} = \lambda_2 \overline{\vec{v}_1}$ since A is a real-valued matrix.

Thus we can take

$$\vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

Example 4. The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle φ of the rotation, where $-\pi < \varphi \leq \pi$, and give the scale factor r.

$$A = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$$

By the general discussion below. we know $a = \sqrt{3}$, $b = -3$

Then $\lambda = \sqrt{3} \pm 3i$. and $r = |\lambda| = \sqrt{a^2+b^2} = \sqrt{(\sqrt{3})^2 + (-3)^2} = \sqrt{12} = 2\sqrt{3}$

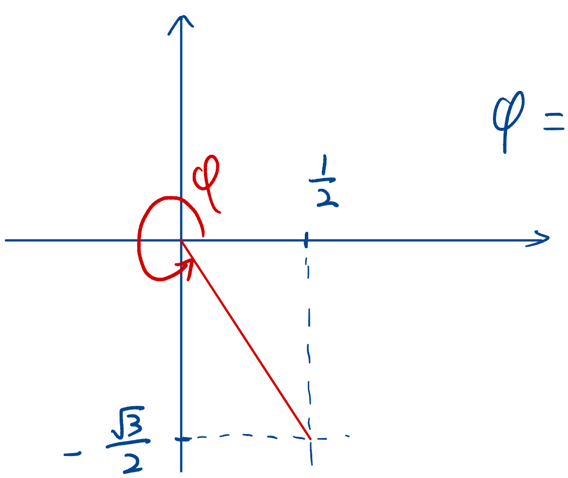
We need to find φ such that

$$\begin{cases} \cos\varphi = \frac{a}{r} = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} \\ \sin\varphi = \frac{b}{r} = \frac{-3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2} \end{cases}$$

From trigonometry.

$$\varphi = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{-3}{\sqrt{3}}\right) = \arctan(-\sqrt{3})$$

$$= -\frac{\pi}{3} \text{ radians.}$$



General Discussion:

- If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real and not both zero, then the eigenvalues of A are $\lambda = a \pm bi$.

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (\lambda - a)^2 + b^2 = 0 \Rightarrow (\lambda - a)^2 = -b^2 \Rightarrow \lambda - a = \pm bi$$
$$\Rightarrow \lambda = a \pm bi$$

- If $r = |\lambda| = \sqrt{a^2 + b^2}$, then

$\quad\; \| \; |a \pm bi| \; //$

$$A = (rI)\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \text{ where } \begin{cases} \cos\varphi = \frac{a}{r} \\ \sin\varphi = \frac{b}{r}. \end{cases}$$

where φ is the angle between the positive x-axis and the ray from $(0, 0)$ through (a, b). See Figure 2.

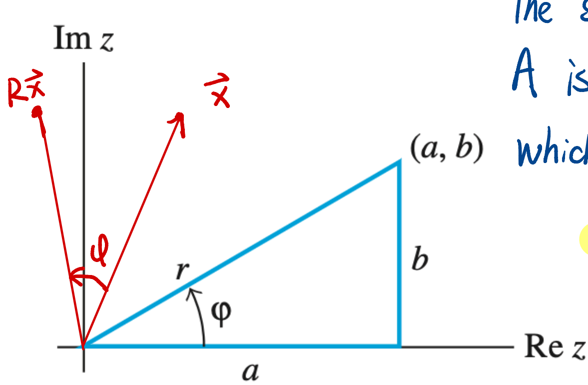


FIGURE 2

The second part of the above factorization of A is a linear transformation $R: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, which is often called the

Rotation Matrix $\quad R = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

It rotates points in xy-plane counterclockwise through an angle φ with respect to the positive x-axis about the origin.

- The angle φ is called the **argument** of $\lambda = a + bi$. Thus the transformation $\mathbf{x} \mapsto A\mathbf{x}$ may be viewed as the composition of a rotation through the angle φ and a scaling by $|\lambda|$. See Figure 3.

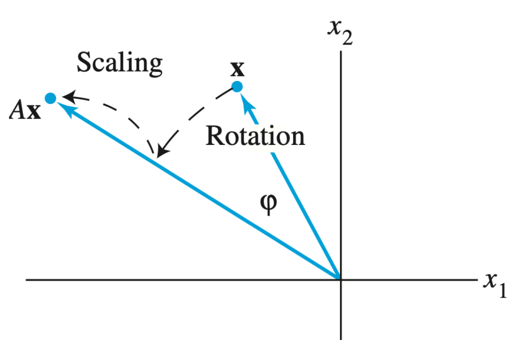


FIGURE 3 A rotation followed by a scaling.

Theorem 9. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi (b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \quad \text{where} \quad P = [\operatorname{Re}\mathbf{v} \quad \operatorname{Im}\mathbf{v}] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Exercise 5. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \overline{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that q is a real number by verifying that $\bar{q} = q$. Give a reason for each step.

$$\bar{q} = \underset{\text{(a)}}{\overline{\overline{\mathbf{x}}^T A\mathbf{x}}} = \underset{\text{(b)}}{\mathbf{x}^T \overline{A\mathbf{x}}} = \underset{\text{(c)}}{\mathbf{x}^T A\overline{\mathbf{x}}} = \underset{\text{(d)}}{(\mathbf{x}^T A\overline{\mathbf{x}})^T} = \underset{\text{(e)}}{\overline{\mathbf{x}}^T A^T \mathbf{x}} = q$$

Solution. (a) properties of conjugates and the fact that $\overline{\mathbf{x}}^T = \overline{\mathbf{x}^T}$
(b) $\overline{A\mathbf{x}} = A\overline{\mathbf{x}}$ and A is real
(c) $\mathbf{x}^T A\overline{\mathbf{x}}$ is a scalar and hence may be viewed as a 1×1 matrix
(d) properties of transposes
(e) $A^T = A$ and the definition of q